

MONODROMY OF PROJECTIVE CURVES

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ABSTRACT. The uniform position principle states that, given an irreducible nondegenerate curve $C \subset \mathbb{P}^r(\mathbb{C})$, a general $(r-2)$ -plane $L \subset \mathbb{P}^r$ is *uniform*, that is, projection from L induces a rational map $C \dashrightarrow \mathbb{P}^1$ whose monodromy group is the full symmetric group. In this paper we first show the locus of non-uniform $(r-2)$ -planes has codimension at least two in the Grassmannian. This result is sharp because, if there is a point $x \in \mathbb{P}^r$ such that projection from x induces a map $C \dashrightarrow \mathbb{P}^{r-1}$ that is not birational onto its image, then the Schubert cycle $\sigma(x)$ of $(r-2)$ -planes through x is contained in the locus of non-uniform subspaces. For a smooth curve C in \mathbb{P}^3 , we show any irreducible surface of non-uniform lines is a Schubert cycle $\sigma(x)$ as above, unless C is a rational curve of degree three, four or six.

1. INTRODUCTION

The monodromy group of a branched covering of smooth complex curves has been object of research since the early days of algebraic geometry. Zariski [27] showed that, for a general smooth complex projective curve X of genus $g > 6$, the monodromy group of any covering $X \rightarrow \mathbb{P}^1$ is not solvable. This result has been greatly generalized recently and it is now known that, for a general curve X of genus $g > 3$, the monodromy group of a degree d indecomposable covering $X \rightarrow \mathbb{P}^1$ is either S_d , in which case we say the covering is *uniform*, or A_d [9]. As far as existence, it is well known that a general (resp. every) curve X of genus g admits a degree d uniform covering $X \rightarrow \mathbb{P}^1$ if $d \geq \frac{g+2}{2}$ (resp. if $d \geq g+2$), while only recently it has been established that a general (resp. every) curve X of genus g admits a degree d covering $X \rightarrow \mathbb{P}^1$ with monodromy group A_d if $d \geq 2g+1$ [17] (resp. if $d \geq 12g+4$ [2]).

Recently some interesting results have appeared on the monodromy groups of coverings obtained projecting a smooth plane curve from a point. Miura and Yoshihara [19, 20, 26] have determined all groups occurring as monodromy groups of projections of smooth plane curves of degree $d \leq 5$. Cukierman [4] has shown that, for a general smooth plane curve $C \subset \mathbb{P}^2$ of degree d and for every point $x \in \mathbb{P}^2 \setminus C$, projection from x induces a *uniform* covering $C \rightarrow \mathbb{P}^1$.

In this paper we begin a systematic study of coverings obtained by projections of curves embedded in projective space. Our starting point is the classical *uniform position principle*.

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Consider a degree d irreducible algebraic curve $C \subset \mathbb{P}^r$ that is not contained in a hyperplane. The uniform position principle, in the formulation of Joe Harris [11], states that, if U is the set of hyperplanes meeting C transversally, then the monodromy group of the covering

$$(1) \quad \pi : \{(x, H) \in C \times U : x \in H\} \rightarrow U, \quad \pi(x, H) = H,$$

is the full symmetric group S_d . This statement is equivalent - by a Lefschetz type theorem on the fundamental group due to Zariski [28] - to the assertion that a *general* $(r-2)$ -plane $L \subset \mathbb{P}^r$ is uniform, that is, projection from L exhibits the normalization \tilde{C} of C as a uniform covering of \mathbb{P}^1 (cf. Proposition 2.4).

Our first result is

Theorem 1.1. *In the Grassmannian $\mathbb{G}(r-2, \mathbb{P}^r)$ the locus of non-uniform subspaces has codimension at least two.*

Note that this theorem strengthens the uniform position principle. For smooth curves it is an easy result (cf. Remark 3.1), but for curves with arbitrary singularities the proof is more involved and is carried out in Section 3, where we also consider subspaces that meet C . In particular, for a curve in \mathbb{P}^2 , our result says only finitely many points of \mathbb{P}^2 , including those lying on C , may fail to be uniform. The theorem is sharp because there are curves with codimension two families of non-uniform subspaces. To see this, observe a subspace L is certainly not uniform if it is *decomposable*, that is, the projection $\pi_L : \tilde{C} \rightarrow \mathbb{P}^1$ factors as the composition of two morphisms of degree at least two. And this is the case if L contains a non birational point x , that is, a point x such that projection from x induces a morphism $\pi_x : \tilde{C} \rightarrow \mathbb{P}^{r-1}$ that is not birational onto its image. Thus the Schubert cycle $\sigma(x)$ of $(r-2)$ -planes through a non birational point x is contained in the non-uniform locus.

In Section 4 we give strong restrictions on families of decomposable $(r-2)$ -planes. For a non rational curve $C \subset \mathbb{P}^3$, we show any codimension two irreducible family of decomposable lines is in fact a Schubert cycle $\sigma(x)$ for some non birational point $x \in \mathbb{P}^3$. More generally, we prove:

Theorem 1.2. *Suppose $C \subset \mathbb{P}^r$ is a nondegenerate irreducible curve, and $\Sigma \subset \mathbb{G}(r-2, \mathbb{P}^r)$ is an irreducible closed subscheme of dimension $r-1$ such that*

- (1) *the general subspace $L \in \Sigma$ does not meet C and is decomposable;*
- (2) *every hyperplane H in \mathbb{P}^r contains at least one subspace $L \in \Sigma$.*

Then C is rational.

In the case of *smooth* curves C in \mathbb{P}^3 , we can actually classify pairs (C, Σ) where Σ is a surface of non-uniform lines for C . The result we prove in Sections 5 and 6 is

Theorem 1.3. *Suppose $C \subset \mathbb{P}^3$ is a smooth irreducible nondegenerate curve, $\Sigma \subset \mathbb{G}(1, \mathbb{P}^3)$ is an irreducible surface, and the general line in Σ does not meet C and is not uniform. Then one of the following three possibilities holds:*

- (1) *either there exists a non birational point $x \in \mathbb{P}^3$ such that Σ is the Schubert cycle $\sigma(x)$ of lines through x ; or*
- (2) *C is a twisted cubic curve, and the general line in Σ is the intersection of two osculating planes to C ; or*

- (3) C is a rational curve of degree 4 (resp. 6), the general line L in Σ is the intersection of two bitangent (resp. tritangent) planes to C , and the projection π_L is decomposable in the form $\tilde{C} \rightarrow \mathbb{P}^1 \xrightarrow{\beta} \mathbb{P}^1$ where β has degree 2.

In fact, every rational curve of degree 3 or 4 admits a surface of non-uniform lines as in the theorem. The rational sextic curve studied in [3] admits a surface of non-uniform lines, but we show this is not the case for a general rational sextic - see Theorem 6.10.

We close the paper, in Section 7, giving examples of one dimensional families of non-uniform lines for curves C of arbitrary genus.

We work over the field \mathbb{C} of complex numbers. If V is a \mathbb{C} -vector space, we denote by $\mathbb{P}(V)$ the projective space of lines in V . The sentence *for a general point x in an algebraic variety X* means for all points x in some Zariski open dense subset of X .

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2. PRELIMINARIES

We collect in this preliminary section some well known results about branched coverings of algebraic curves. For the convenience of the reader, we include proofs of some of these results.

We begin recalling the definition of the *Galois group* or *monodromy group* of a finite morphism of algebraic varieties [10]. Suppose X and Y are irreducible algebraic varieties of the same dimension over \mathbb{C} , and $\pi : X \rightarrow Y$ is a generically finite dominant morphism of degree d . Then the function field $K(X)$ is a finite algebraic extension of $K(Y)$ of degree d . Let \hat{K} denote a Galois closure of $K(X)/K(Y)$. We define the Galois group $G = G_\pi$ of the morphism $\pi : X \rightarrow Y$ to be the Galois group of the extension $\hat{K}/K(Y)$: the group G consists of the automorphisms of the field \hat{K} that leave every element of $K(Y)$ fixed. Given a general point $q \in Y$, the group G acts faithfully on the fibre $\pi^{-1}(q)$ and thus may be regarded as a subgroup of $\text{Aut}(\pi^{-1}(q)) \cong S_d$, and as such it is a transitive subgroup. There is an equivalent, more geometric description of the Galois group. We can choose a Zariski open dense subset $U \subset Y$ such that the restriction of π to $\pi^{-1}(U) \rightarrow U$ is étale, thus a d -sheeted covering in the classical topology. If q is a point of U , lifting loops at q to $\pi^{-1}(U)$ we obtain the monodromy representation $\rho : \pi_1(U, q) \rightarrow \text{Aut}(\pi^{-1}(q))$. It is not difficult to see that the image of ρ , that is, the monodromy group of the covering, is isomorphic to the Galois group G , and is therefore independent of the choice of U [10, Proposition p. 689].

Definition 2.1. Let $\pi : X \rightarrow Y$ be a generically finite dominant morphism of degree d between complex algebraic varieties. We say that π is *uniform* if the monodromy group of π is the full symmetric group S_d . We say that π is *decomposable* if there exists an open dense subset $U \subseteq Y$ over which π factors as

$$\pi^{-1}(U) \xrightarrow{\alpha} V \xrightarrow{\beta} U$$

where α and β are finite morphisms of degree at least 2.

Remark 2.2. One knows π is decomposable if and only if there is an intermediate field in the extension $K(X)/K(Y)$, and this is equivalent to the condition the Galois group G_π be imprimitive. Furthermore:

- (1) If π is uniform, then it is indecomposable.
- (2) If π is indecomposable and G_π contains a transposition, then π is uniform.

The first statement is clear. Suppose $G_\pi \subset S_d$ contains a transposition, and let $N \subseteq G_\pi$ be the subgroup generated by the transpositions contained in G_π . Then N is a nontrivial normal subgroup of G . If N is not transitive, then G_π is imprimitive and π is decomposable. On the other hand, if N is transitive, then π is uniform because a transitive subgroup of S_d generated by transpositions must be all of S_d - see for example [4].

Throughout the paper C denotes an irreducible reduced curve of degree d in \mathbb{P}^r - except in the Section 6 where we treat rational curves of even degree $2d$. We assume C is nondegenerate, that is, not contained in a hyperplane. We denote by \tilde{C} the normalization of C , by $f : \tilde{C} \rightarrow \mathbb{P}^r$ the normalization map followed by the embedding of C in \mathbb{P}^r , and by $|V|$ the base point free g_d^r on \tilde{C} corresponding to f . We have $|V| = \mathbb{P}(V)$ where $V \subseteq H^0(\tilde{C}, f^*\mathcal{O}_{\mathbb{P}^r}(1))$ is the subspace generated by the restrictions of the linear forms on \mathbb{P}^r to \tilde{C} . There is a one to one correspondence between hyperplanes H in $\mathbb{P}^r = \mathbb{P}(V^*)$ and divisors of the linear series $|V|$, and we let $D_H = f^*(H)$ denote the divisor corresponding to H . If H is transversal to C , then D_H may be identified with the intersection of C and H .

Given a linear subspace $M \subset \mathbb{P}^r$ of codimension $s \geq 1$, projection from M is the morphism $\pi_M : \mathbb{P}^r \setminus M \rightarrow \mathbb{P}_M^{s-1}$ mapping a point $x \notin M$ to the linear subspace generated by x and M , where $\mathbb{P}_M^{s-1} \cong \mathbb{P}^{s-1}$ denotes the Schubert cycle of codimension $s-1$ subspaces of \mathbb{P}^r containing M . Projection from M induces a morphism $\pi_M : \tilde{C} \rightarrow \mathbb{P}^{s-1}$.

For a codimension two subspace L , the morphism $\pi_L : \tilde{C} \rightarrow \mathbb{P}_L^1$ is finite (because C is nondegenerate) of degree $d_L = d - \deg(D_L)$, where D_L is the base locus of the linear system $\{D_H : H \supset L\}$. We let G_L denote the monodromy group of π_L . We say L is *uniform* (resp. *decomposable*) if the morphism π is uniform (resp. decomposable).

Remark 2.3. Let $\mathbb{G}(r-2, \mathbb{P}^r)$ the Grassmannian of codimension two subspaces of \mathbb{P}^r . The set of uniform subspaces (with respect to a given curve C) is constructible in $\mathbb{G}(r-2, \mathbb{P}^r)$, and it contains a non-empty open set (cf. 2.4 below).

The projections π_L , as L varies in the set of $(r-2)$ -planes that do not meet C , are restrictions of the global projection:

$$\pi : I = \{(P, H) \in \tilde{C} \times \mathbb{P}^{r*} : f(P) \in H\} \rightarrow \mathbb{P}^{r*}, \quad (P, H) \mapsto H.$$

Note that I is a \mathbb{P}^{r-1} -bundle over \tilde{C} . In fact, from the Euler sequence one sees I is the projective bundle of *lines* in the fibres of $f^*\Omega_{\mathbb{P}^r}(1)$. By construction the scheme theoretic fibre of π over $H \in \mathbb{P}^{r*}$ is $f^*(H) = D_H$; thus π is a finite flat surjective morphism (= a branched covering) of degree $d = \deg(C)$.

We use the well known fact [1, p.111] that the monodromy of $\pi : I \rightarrow \mathbb{P}^{r*}$ is the full symmetric group S_d to show the general codimension two subspace L is uniform - for smooth curves this is proven for example in [4]. More precisely we have:

Proposition 2.4. *Let $L \subset \mathbb{P}^r$ be a codimension two subspace that does not meet C , and let B_{red} denote the branch divisor of $\pi : I \rightarrow \mathbb{P}^{r*}$ taken with its reduced scheme structure. If the line \mathbb{P}_L^1 meets B_{red} transversally, then L is uniform.*

Proof. Given a codimension two subspace L , the codomain of the projection π_L is the line $\mathbb{P}_L^1 \subset \mathbb{P}^{r*}$. If L does not meet C , then $\pi^{-1}(\mathbb{P}_L^1) \cong \tilde{C}$, and under this isomorphism the restriction of π to $\pi^{-1}(\mathbb{P}_L^1)$ corresponds to π_L .

The monodromy group of $\pi : I \rightarrow \mathbb{P}^{r*}$ is the full symmetric group [1, Lemma p.111]. Since \mathbb{P}_L^1 meets B_{red} transversally, by a Lefschetz-type theorem due to Zariski [28] (see also [8, p. 34] and [25]), the inclusion $\mathbb{P}_L^1 \hookrightarrow \mathbb{P}^{r*}$ induces an epimorphism $\pi_1(U \cap \mathbb{P}_L^1) \rightarrow \pi_1(U)$, where $U = \mathbb{P}^{r*} \setminus B_{red}$. Hence the monodromy of π_L is the full symmetric group as well. \square

Remark 2.5. We will see below the branch divisor B contains the dual variety C^* whose points are the hyperplanes tangent to C . We recall the definition of C^* and the biduality theorem. Let $W_C^0 \subset \mathbb{P}^r \times \mathbb{P}^{r*}$ denote the set of pairs (x, H) where x is a smooth point of C and H is a hyperplane containing the tangent line $T_x C$, and let W_C denote the closure of W_C^0 . We say that a hyperplane H is tangent to C at a (not necessarily smooth) point x if the pair (x, H) belongs to W_C . The dual variety $C^* \subset \mathbb{P}^{r*}$ is defined as the set of hyperplanes that are tangent to C , i.e., as the projection of W_C in \mathbb{P}^{r*} . Of course, these definitions apply to any subvariety of \mathbb{P}^r . The biduality theorem asserts $W_C = W_{C^*}$, that is, a hyperplane H is tangent to C at x if and only if in the dual projective space the hyperplane x^* is tangent to C^* at the point H^* .

Remark 2.6. In fact, the hypothesis L does not meet C is redundant in Proposition 2.4: one can check, for example using the biduality theorem, that the fact \mathbb{P}_L^1 meets B_{red} transversally implies L does not meet C .

We now wish to describe the branch divisor B of π , and the subspaces L meeting B_{red} non transversally. For this we need to introduce the *ramification sequence* of the linear series $|V|$ at a point $P \in \tilde{C}$ as defined in [1]. Since V is $r + 1$ -dimensional, for a given $P \in \tilde{C}$ the "multiplicity at P function" $H \mapsto m_P(D_H)$ takes on exactly $r + 1$ -values, so there are nonnegative integers $\alpha_0(P) \leq \alpha_1(P) \leq \dots \leq \alpha_r(P)$ such that

$$\{m_P(D_H) : H^* \in \mathbb{P}(V)\} = \{\alpha_0, 1 + \alpha_1, 2 + \alpha_2, \dots, r + \alpha_r\}.$$

For each $k = 0, 1, \dots, r - 1$ the set of hyperplanes H with

$$m_P(D_H) \geq k + 1 + \alpha_{k+1}(P)$$

is a linear subspace $M_{k,P}^*$ of $\mathbb{P}(V)$ of codimension $k + 1$. Its dual $f_k(P)$ is a k -dimensional subspace of $\mathbb{P}(V^*) = \mathbb{P}^r$ called the osculating k -plane of C at P - of course, $f_1(P) = T_P C$ is the tangent line to the branch of C through P . The osculating k -plane is the unique k -plane M such that

$$m_P(D_M) = k + 1 + \alpha_{k+1}(P)$$

(the other k -planes have smaller multiplicity at P).

One easily verifies H is tangent to C at x if and only if there is $P \in \tilde{C}$ such that $x = f(P)$ and $T_P C \subset H$. Thus $H \in C^*$ if and only if there is $P \in \tilde{C}$ such that $m_P(D_H) \geq 2 + \alpha_2(P)$.

We can now describe the branch locus B of $\pi : I \rightarrow \mathbb{P}^{r*}$. For each point $x \in C$ we let

$$\beta(x) = \sum_{\{P \in \tilde{C} : f(P) = x\}} \alpha_1(P)$$

denote the total ramification of f at x .

Proposition 2.7. *Let $C^* \subset \mathbb{P}^{r*}$ denote the hypersurface dual to C , and for each point $x \in \mathbb{P}^r$ let $\Pi_x \subset \mathbb{P}^{r*}$ denote the hyperplane dual to x . Let B denote the branch divisor of $\pi : I \rightarrow \mathbb{P}^{r*}$. Then in the group of Weil divisors of \mathbb{P}^{r*} we have:*

$$B = C^* + \sum_{\{x \in C\}} \beta(x) \Pi_x.$$

Furthermore, the multiplicity of B at a point $H^* \in \mathbb{P}^{r*}$ is

$$m_{H^*}(B) = \sum_{\{P \in D_H\}} (m_P(D_H) - 1)$$

Proof. The ramification divisor R of π is defined as the scheme of zeros on I of the Jacobian determinant of π , and the branch divisor $B = \pi_*(R)$ as the push forward of R to \mathbb{P}^{r*} . Equivalently, R is defined by the zeroth Fitting ideal of the relative sheaf of differential $\Omega_{I/\mathbb{P}^{r*}}$ [15]. This definition is functorial, as is the push forward of divisors; it follows that, if $\mathbb{P}_L^1 \subset \mathbb{P}^{r*}$ is a line corresponding to a codimension two subspace L not meeting C , the branch divisor B_L of $\pi_L : \pi^{-1}(\mathbb{P}_L^1) \cong \tilde{C} \rightarrow \mathbb{P}_L^1$ is the scheme theoretic intersection of B and \mathbb{P}_L^1 . Since $D_H = \sum_{P \in \tilde{C}} m_P(D_H)P$ is the fibre of π_L over H^* and we work over \mathbb{C} , the branch divisor of π_L is

$$B_L = \sum_{\{H^* \in \mathbb{P}_L^1\}} \left(\sum_{\{P \in D_H\}} (m_P(D_H) - 1) \right) H^*.$$

Now the multiplicity of B at H^* is precisely the multiplicity of intersection at H^* of B with a general line through H^* , hence we conclude

$$m_{H^*}(B) = \sum_{\{P \in D_H\}} (m_P(D_H) - 1).$$

If H is a tangent hyperplane to C , then there is a point $P \in \tilde{C}$ appearing with multiplicity at least two in D_H , thus H belongs to B and C^* is a component of B . Furthermore, since we are in characteristic zero, the general tangent plane H to C is simply tangent, that is, $m_{H^*}(B^*) = 1$. Thus B is generically smooth along C^* , which means that C^* is a reduced component of B .

On the other hand, suppose H is in the branch locus but not in C^* . Then in the fibre D_H there is a point P such that $m_P(D_H) \geq 2$ because H is in the branch locus and $m_P(D_H) = 1 + \alpha_1(P)$ because H is not tangent to C . Therefore $1 + \alpha_1(P) \geq 2$ so that $m_P(D_K) \geq 2$ for every hyperplane K through $x = f(P)$. Hence the entire hyperplane $\Pi_x \subset \mathbb{P}^{r*}$ is contained in the branch locus. To compute the multiplicity of Π_x in B , we pick a hyperplane H through x that is not tangent to C and does not pass through any

other singularity of f , i.e. Π_x is the only component of B through H^* . Then the desired multiplicity is equal to:

$$m_{H^*}(B) = \sum_{\{P \in D_H\}} (m_P(D_H) - 1) = \sum_{\{P \in \tilde{C}: f(P)=x\}} \alpha_1(P) = \beta(x).$$

Note: since $x \in C$, by biduality Π_x is tangent to C^* . \square

We can now list the codimension two subspaces L for which \mathbb{P}_L^1 does not meet B_{red} transversally:

Lemma 2.8. *Fix a codimension two subspace $L \subset \mathbb{P}^r$ not meeting C . Let Π_1, \dots, Π_n be the hyperplanes of \mathbb{P}^{r*} contained in B . Then \mathbb{P}_L^1 meets B_{red} transversally unless*

- (1) L is contained in a hyperplane H with $H^* \in C_{sing}^*$, or
- (2) L is contained in a hyperplane H with $H^* \in C^* \cap \Pi_i$, or
- (3) L is contained in a hyperplane H with $H^* \in \Pi_i \cap \Pi_j$.

Proof. The subspace L satisfies one of the conditions (1),(2),(3) if and only if \mathbb{P}_L^1 meets the singular locus of B_{red} . Thus we only have to check that if \mathbb{P}_L^1 is tangent to B_{red} at a smooth point H^* , then L meets C . This follows from biduality if $H^* \in C^*$, while if $H^* \in \Pi_i = \Pi_{x_i}$ for some i , then \mathbb{P}_L^1 is tangent to and hence contained in Π_{x_i} , so that $x_i \in L \cap C$. \square

We close this preliminary section recalling the relation between the monodromy group G_L and the ramification type of π_L .

Proposition 2.9. *Let $\tau : \tilde{C} \rightarrow \mathbb{P}^1$ be a finite morphism of degree d , and let $H_1, \dots, H_n \in \mathbb{P}^1$ be the branch points of τ . Then to each H_i one can associate a permutation σ_i in the Galois group $G_\tau \subseteq S_d$ so that*

- (1) *the Galois group G_τ is generated by any set of $n-1$ permutations among the σ_i ;*
- (2) *if $\sum_{j=1}^{t_i} m_{ij} P_{ij}$ denotes the scheme theoretic fibre of τ over H_i , then σ_i is a permutation with cycle structure $(m_{i1}, \dots, m_{it_i})$. Thus, if H_i is a simple branch point, the permutation σ_i is a transposition.*

In particular, if all but one of the branch points are simple, the morphism π is uniform.

Proof. This is classical and explained for example in [18, §III.4]. It follows from the fact the fundamental group of $\mathbb{P}^1 \setminus \{H_1, \dots, H_n\}$ is generated by small loops $\gamma_1, \dots, \gamma_n$ around H_1, \dots, H_n respectively, with the relation $\gamma_1 \gamma_2 \cdots \gamma_n = 1$. If all the branch points but one are simple, the Galois group is generated by transpositions, hence must be all of S_d . \square

Example 2.10. Let $C \subset \mathbb{P}^3$ be a twisted cubic curve, and for every $P \in C$ let $H(P)$ denote the osculating plane to C at P , so that $D_{H(P)} = 3P$. Given two distinct points $P, Q \in C$, let $L = H(P) \cap H(Q)$. Since $H(P)$ meets C only at P , the line L does not meet C , and the projection π_L has degree 3. By Riemann-Hurwitz, the projection π_L is ramified only over $H(P)$ and $H(Q)$, and by Lemma 2.9 the monodromy group of π_L is A_3 . Letting P and Q vary we find a two dimensional irreducible family of lines L that are non-uniform and indecomposable. We will later show (Theorem 5.2) there is no other smooth curve in \mathbb{P}^3 admitting a two dimensional family of non-uniform indecomposable lines.

If we let P and Q come together, the line L becomes tangent to C , thus π_L is an isomorphism and L is uniform according to our definition.

3. THE LOCUS OF NON-UNIFORM SUBSPACES HAS CODIMENSION AT LEAST TWO

In this section we prove the locus of non-uniform subspaces has codimension at least two in the Grassmannian $\mathbb{G}(r-2, \mathbb{P}^r)$. One easily reduces this statement to the case $r = 2$, and so the problem is to show a plane curve has only finitely many non-uniform points.

Remark 3.1. If C is smooth and $\mathcal{N}(C)$ denotes the locus of codimension two subspaces that do not meet C and are not uniform, the fact that $\mathcal{N}(C)$ has codimension at least two is an immediate consequence of Propositions 2.7 and 2.9. More generally suppose C is a curve for which the branch divisor B of the morphism $\pi : I \rightarrow \mathbb{P}^{r*}$ is reduced, and consider a subspace L in $\mathcal{N}(C)$. Then by Proposition 2.9 there are at least two branch points H_1 and H_2 of π_L that are not simple. By Proposition 2.7 the hyperplanes H_1 and H_2 are singular points of the branch divisor B . Hence \mathbb{P}_L^1 meets the singular locus B_{sing} of the branch divisor in at least two distinct points. Therefore, if \mathcal{N}^* denote the locus of lines $\{\mathbb{P}_L^1 : L \in \mathcal{N}(C)\}$, we have

$$\dim \mathcal{N}(C) = \dim \mathcal{N}^* \leq 2 \dim B_{\text{sing}} \leq 2r - 4,$$

thus $\mathcal{N}(C)$ has codimension at least two in the Grassmannian.

We now generalize the above statement to curves C with arbitrary singularities, without excluding subspaces that meet C . The key case is that of plane curves. When $r = 2$, a codimension two subspace is a point $x \in \mathbb{P}^2$, and $\mathbb{P}_x^1 \subset \mathbb{P}^{2*}$ is the pencil of lines through x . We first consider the case $x \notin C$ (outer points in the terminology of [26]). Proposition 2.4 implies that x is uniform, unless x belongs to one of the finitely many lines listed in 2.8. Thus, if we can show that the general point of an arbitrary line is uniform, we can conclude that there are only finitely many non-uniform outer points. We will need the following lemma:

Lemma 3.2. *Let $\tau : X \rightarrow Y$ be a surjective étale morphism of smooth algebraic varieties, and let $h : Y \rightarrow U$ be a dominant morphism. Assume that, for the general point $q \in U$, the fibres $X_q = (h \circ \tau)^{-1}(q)$ and $Y_q = h^{-1}(q)$ are irreducible. Then, for a general point $q \in U$, the monodromy group of $\tau_q : X_q \rightarrow Y_q$ is a normal subgroup of the monodromy group of $\tau : X \rightarrow Y$.*

Proof. By generic smoothness (cf. [21, Lemma 1.5a])) there is a dense Zariski open subset $U_0 \subset U$ such that $h : h^{-1}(U_0) \rightarrow U_0$ is a fibre bundle in the analytic topology. Thus we may assume that $h : Y \rightarrow U$ is a fibre bundle in the analytic topology. Let $g : Y_q \rightarrow Y$ be the inclusion of the fibre over $q \in U$. The long homotopy sequence of the fibration $h : Y \rightarrow U$ shows that $g_*\pi_1(Y_q)$ is a normal subgroup of $\pi_1(Y)$.

Now, fixed a base point $y \in Y_q$, the monodromy representations of τ and τ_q are related by the commutative diagram:

$$(2) \quad \begin{array}{ccc} \pi_1(Y_q, y) & \xrightarrow{\rho(\tau_q, y)} & \text{Aut}(\tau_q^{-1}(y)) \\ \downarrow g_* & & \downarrow \\ \pi_1(Y, y) & \xrightarrow{\rho(\tau, y)} & \text{Aut}(\tau^{-1}(y)) \end{array}$$

where the vertical arrow on the right is an isomorphism. Since $g_*\pi_1(Y_q)$ is a normal subgroup of $\pi_1(Y)$, the diagram above shows the monodromy group $\text{Im}(\rho(\tau_q, y))$ of τ_q is a normal subgroup of the monodromy group $\text{Im}(\rho(\tau, y))$ of τ . \square

Proposition 3.3. *Let M be a line in \mathbb{P}^2 . Then all but finitely many points $x \in M$ are uniform.*

Proof. Fix a general point q in M . We prove that the monodromy group G_q of π_q is the full symmetric group S_d showing it contains a transposition and it is a normal subgroup of S_d . To see G_q contains a transposition is immediate: it is enough to observe a general point q of M lies on a simple tangent line to C .

To see G_q is normal, we use the previous lemma. For this, we need to consider the map $\pi : I \rightarrow \mathbb{P}^{2*}$ of Proposition 2.4 as a family of projections indexed by the points of M . To this end we fix a second line L , let z_0 be the point of intersection of L and M , and write $L' = L \setminus \{z_0\}$, $M' = M \setminus \{z_0\}$. Then the map $L' \times M' \rightarrow \mathbb{P}^{2*}$ sending a pair (t, u) to the line joining them is an open embedding. Let Y be the open subscheme $L' \times M'$ of \mathbb{P}^{2*} and define

$$X = \pi^{-1}(L' \times M'), \quad \tau = \pi|_X : X \rightarrow Y.$$

Since Y is open in \mathbb{P}^{2*} , the monodromy group of τ coincides with the monodromy group of π , i.e., it is the symmetric group S_d . Now we restrict Y to an open subset over which τ is étale, and apply Lemma 3.2 to τ with $U = M'$ to conclude that, for general $q \in M'$, the monodromy group of $\tau_q : X_q \rightarrow Y_q$ is normal in S_d . Now it is immediate to verify that X_q is isomorphic to an open subscheme of \tilde{C} , and with this identification $\tau_q : X_q \rightarrow Y_q$ is projection from q . Thus the monodromy group G_q is normal in S_d . This finishes the proof. \square

Remark 3.4. The same argument shows that, for a nondegenerate curve $C \subset \mathbb{P}^r$, given any hyperplane M in \mathbb{P}^r , the general codimension two subspace L contained in M is uniform.

We can also show, with an argument similar to the proof of 3.3, that only finitely many points of C (inner points) may fail to be uniform:

Proposition 3.5. *A general point of C is uniform.*

Proof. Fix a general point q in C and a general line H . The fibre of π_q over H is $C \cap H$ with the point q removed, therefore the monodromy group G_q is a subgroup of S_{d-1} . We may assume $d = \deg(C) \geq 3$. Then we can find a simple tangent to C passing with multiplicity one through q . It follows the monodromy of π_q contains a transposition. Thus it is enough to show the monodromy group G_q is normal in S_{d-1} .

To this end, fix a line L , and let C' denote the set of smooth points of C that do not belong to L ; similarly let L' denote the set of points of L that do not lie on C . Then $L' \times C'$ is an open subscheme of the incidence correspondence

$$I = \{(P, H) \in \tilde{C} \times \mathbb{P}^{2*} : f(P) \in H\}$$

via the embedding $(t, x) \mapsto (f^{-1}(x), \overline{xt})$. The projection $\pi : L' \times C' \subset I \rightarrow \mathbb{P}^{2*}$ maps a pair (t, x) to the line \overline{xt} , and of course we know π is generically finite of degree d with monodromy group S_d .

In order to be able to apply Lemma 3.2 we need to pull back the covering π to $L' \times C'$. This pull back, that is the fibred product

$$T = (L' \times C') \times_{\mathbb{P}^{2*}} (L' \times C'),$$

has two components: one is the diagonal, the other is isomorphic to

$$Z = \{(x, y) \in C' \times C' : x \neq y; \overline{xy} \cap L \in L'\}.$$

Let $\pi_Z : Z \rightarrow L' \times C'$ denote the composition of the inclusion $Z \hookrightarrow T$ with the projection $pr_1 : T \rightarrow L' \times C'$, that is, $\pi_Z(x, y) = (\overline{xy} \cap L, y)$. Then $\pi_Z : Z \rightarrow L' \times C'$ is generically finite of degree $d - 1$, and, for general $q \in C$, the restriction $\pi_q : Z_q \rightarrow L' \times q$ is (modulo obvious identifications) projection from q . Using Lemma 3.2 we conclude the monodromy group G_q is a normal subgroup of G_{π_Z} , and we will be done if we can show $G_{\pi_Z} = S_{d-1}$.

Choose an open dense subset $Y \subset \mathbb{P}^{2*}$ such that

$$\pi|_{\pi^{-1}(Y)} : W = \pi^{-1}(Y) \rightarrow Y$$

is étale, and fix a base point $w \in W$. Then as in 3.2 we look at the diagram:

$$(3) \quad \begin{array}{ccc} \pi_1(W, w) & \xrightarrow{\rho(\pi_Z)} & \text{Aut}(\pi_Z^{-1}(w)) \cong S_{d-1} \\ \downarrow \pi_* & & \downarrow \\ \pi_1(Y, \pi(w)) & \xrightarrow{\rho(\pi)} & \text{Aut}(\pi^{-1}(\pi(w))) \cong S_d \end{array}$$

Since $W \rightarrow Y$ is a topological covering, the image of the vertical map π_* is the stabilizer of w under the action of $\pi_1(Y, \pi(w))$ on the fibre $\pi^{-1}(\pi(w))$. The bottom row of the diagram is surjective because $G_\pi = S_d$. Therefore the image of $\rho(\pi) \circ \pi_*$ is the stabilizer of w in $\text{Aut}(\pi^{-1}(\pi(w)))$, which is also the image of the vertical map on the right. Therefore $\rho(\pi_Z)$ is surjective, that is, the monodromy group of π_Z is S_{d-1} , and this concludes the proof. \square

We can now prove:

Theorem 3.6. *Let $C \subset \mathbb{P}^r$ be an irreducible non degenerate curve, $r \geq 2$. In the Grassmannian $\mathbb{G}(r-2, \mathbb{P}^r)$ the locus of non-uniform subspaces has codimension at least two.*

Proof. Suppose first $r = 2$. We have to show all but finitely many points $x \in \mathbb{P}^2$ are uniform for C . By Proposition 2.4, if x is non uniform, then x lies either on one of the finitely many lines listed in Lemma 2.8 or on C . By Propositions 3.3 and 3.5 only finitely many such points may fail to be uniform.

Now assume $r \geq 3$. Fix an irreducible nontrivial effective divisor $\mathcal{D} \subset \mathbb{G}(r-2, \mathbb{P}^r)$. We show the general subspace L in \mathcal{D} is uniform reducing to the case $r = 2$. For this, let M be a general codimension three subspace, so that projection from M

$$\pi_M : \tilde{C} \rightarrow \mathbb{P}_M^2$$

is a birational morphism of \tilde{C} onto $C_1 = \pi_M(\tilde{C})$. Then projecting C from a codimension two subspace L containing M is the same as projecting C_1 from the point $\pi_M(L)$ of \mathbb{P}_M^2 . We have seen there are only finitely many non-uniform points in \mathbb{P}_M^2 , thus we see there are only finitely many codimension two subspaces L that contain M and are not uniform.

To finish, we need to observe *a general codimension three subspace M of \mathbb{P}^3 is contained in infinitely many general subspaces L of \mathcal{D}* . More precisely, let $\sigma(M) \cong \mathbb{P}^2$ denote the set of codimension two subspaces L that contain a given M , and fix a proper subvariety $\Sigma \subset \mathcal{D}$. The set of codimension three subspaces M such that

$$\dim \sigma(M) \cap (\mathcal{D} \setminus \Sigma) > 0$$

contains a dense open subset $U \subset \mathbb{G}(r-3, \mathbb{P}^r)$. Indeed, since every nontrivial effective divisor on the Grassmannian is ample, the dimension of $\mathcal{D} \cap \sigma(M)$ is at least one for every codimension three subspace M . On the other hand, a dimension count on the incidence variety $\{(M, L) : M \subset L\}$ shows that, for M general in $\mathbb{G}(r-3, \mathbb{P}^r)$, the intersection $\Sigma \cap \sigma(M)$ is at most zero dimensional. \square

Remark 3.7. The bound on the codimension of the family of non-uniform subspaces is sharp. Indeed, there are curves C for which there exist points $x \in \mathbb{P}^r$ giving a non birational projection. Fix such a point x . If L is a codimension two subspace containing x , then π_L factors through π_x hence L is decomposable and in particular it is non-uniform. Thus every codimension two subspace L through x is non-uniform, and the family of such subspaces has codimension two in the Grassmannian.

4. FAMILIES OF DECOMPOSABLE SUBSPACES

In this section we show that, if $C \subset \mathbb{P}^r$ is not rational, an irreducible family Σ of decomposable $(r-2)$ -planes either has dimension less than $r-1$ or is special in the sense that a general hyperplane of \mathbb{P}^r does not contain any subspace in Σ . More precisely:

Theorem 4.1. *Suppose $\Sigma \subset \mathbb{G}(r-2, \mathbb{P}^r)$ is an irreducible closed subscheme of dimension $r-1$ such that*

- (1) *the general subspace $L \in \Sigma$ does not meet C and is decomposable;*
- (2) *every hyperplane H in \mathbb{P}^r contains at least one subspace $L \in \Sigma$.*

Then C is rational.

Proof. We give now a sketch of the proof, the rest of the section being dedicated to filling in the details. A classical theorem of de Franchis [5] states that there are only finitely many curves Y (up to isomorphisms) for which there exists a surjective morphism $\tilde{C} \rightarrow Y$. Since the projections π_L form a continuous family, it follows we can find a curve Y and an integer e such that, for every L in an open subset W of Σ , the projection π_L factors through some degree e morphism $\alpha : \tilde{C} \rightarrow Y$, where α varies in an irreducible family of morphisms \mathcal{M} .

Now by hypothesis a general hyperplane H contains at least one subspace $L \in W$, hence $C \cap H$ contains a subset of e points that make up a fibre of some morphism $\alpha \in \mathcal{M}$. By the uniform position principle, any set of e points in $C \cap H$ is a fibre of some $\alpha \in \mathcal{M}$.

If $g(Y) \geq 1$, then the set of morphisms $\tilde{C} \rightarrow Y$ of degree e is finite (up to translations in Y if $g(Y) = 1$), hence we can find a single morphism α such that the projections π_L with $L \in W$ all factor through α . But then not every set of e points in $C \cap H$ is a fibre of α , a contradiction. Thus $Y \cong \mathbb{P}^1$.

It follows that any two fibres of a morphism $\alpha : \tilde{C} \rightarrow Y \cong \mathbb{P}^1$ are linearly equivalent as divisors on \tilde{C} . Now the uniform position principle implies that in $C \cap H$ any two disjoint subsets of cardinality e are linearly equivalent as divisors on \tilde{C} . From this one deduces $2P$ is linearly equivalent to $2Q$ for two general points $P, Q \in \tilde{C}$, hence C is rational. \square

We now fill in the details of the proof.

Lemma 4.2. *Suppose T is an irreducible variety and $\pi : \tilde{C} \times T \rightarrow \mathbb{P}^1 \times T$ is a T -morphism such that for every $t \in T$ the restriction $\pi_t : \tilde{C} \times \{t\} \rightarrow \mathbb{P}^1 \times \{t\}$ is a decomposable finite morphism of degree d .*

Then there exist an open dense subset $W \subset T$, a curve Y , an integer e , $2 \leq e \leq d/2$, and an irreducible family \mathcal{M} of degree e morphisms from \tilde{C} to Y with the following property: for every $t \in W$, there exists $\alpha \in \mathcal{M}$ such that π_t factors through α .

Proof. A well known theorem by de Franchis [5, 14, 23] states that, up to isomorphisms, there are only finitely many smooth projective curves Y , for which there exists a surjective morphism $\alpha : \tilde{C} \rightarrow Y$. Let $\text{Mor}_d(\tilde{C}, \mathbb{P}^1)$ denote the quasiprojective scheme that parametrizes the set of finite morphisms $\tilde{C} \rightarrow \mathbb{P}^1$ of degree d (see [16, I.1.10] or [6]); by the universal property of $\text{Mor}_d(\tilde{C}, \mathbb{P}^1)$ the assignment $t \mapsto \pi_t$ defines a morphism $\psi : T \rightarrow \text{Mor}_d(\tilde{C}, \mathbb{P}^1)$, and by assumption the image $\psi(T)$ is contained in the set of decomposable morphisms. Now by de Franchis Theorem the set of decomposable morphisms in $\text{Mor}_d(\tilde{C}, \mathbb{P}^1)$ is covered by finitely many sets, images of maps

$$\text{Mor}_{e_j}(\tilde{C}, Y_i) \times \text{Mor}_{d/e_j}(Y_i, \mathbb{P}^1) \rightarrow \text{Mor}_d(\tilde{C}, \mathbb{P}^1), \quad (\alpha, \beta) \mapsto \beta \circ \alpha.$$

where Y_i varies in a finite set of curves, and e_j in the set of integers > 1 that divide d . Since T is irreducible, there is one of these sets that contains an open dense subset $A \subset \psi(T)$. Then we can take $W = \psi^{-1}(A)$, and the rest is clear. \square

We will use the following version of the

Monodromy Theorem 4.3 ([1] pp. 111-2). *Let $C \subset \mathbb{P}^r$ be a reduced irreducible curve of degree d , and denote by $U \subset \mathbb{P}^{r*}$ the open set of hyperplanes transverse to C . Then*

$$I(d) = \{(P_1, \dots, P_d, H) : H \in U; \text{the } P_i \text{'s are distinct and } D_H = P_1 + \dots + P_d\}$$

is an irreducible variety.

Note that the symmetric group S_d acts on the covering projection $p : I(d) \rightarrow U$, and U is the quotient of $I(d)$ modulo the action of S_d . It follows that for any subgroup G of S_d , the quotient space $I(d)/G$ is an irreducible variety. In particular, we have:

Corollary 4.4. *For every positive integer m , let \tilde{C}_m denote the m -th symmetric product of \tilde{C} , which we identify with the set of effective divisors of degree m on \tilde{C} . Then the sets*

(1)

$$I_e = \{(E, H) \in \tilde{C}_e \times U : E \leq D_H\}$$

(2)

$$I_{e,f} = \{(E, F, H) \in \tilde{C}_e \times \tilde{C}_f \times U : E + F \leq D_H\}$$

are irreducible.

Proof. I_e is the quotient of $I(d)$ by the subgroup of S_d that stabilizes the subset $\{1, \dots, e\}$ of $\{1, \dots, d\}$; while $I_{e,f}$ is the quotient of $I(d)$ by the subgroup of S_d that stabilizes the subsets $\{1, \dots, e\}$ and $\{e+1, \dots, e+f\}$ of $\{1, \dots, d\}$. \square

Proof of Theorem 4.1. Note that by assumption (2) the morphism

$$p : J = \{(L, H) \in \Sigma \times \mathbb{P}^{r*} : L \subset H\} \rightarrow \mathbb{P}^{r*}, \quad (L, H) \mapsto H$$

is dominant and generically finite. It follows that every open dense subset $T \subset \Sigma$ has the property that, given a general hyperplane H , we can find L in T that is contained in H .

Thus under the hypotheses of the theorem we can find $T \subset \mathbb{G}(r-2, \mathbb{P}^r)$ irreducible of dimension $r-1$ such that, for every $L \in T$, we have $L \cap C = \emptyset$, the projection π_L is decomposable, and the general hyperplane H contains some $L \in T$. We wish to apply Lemma 4.2. For this, we fix a line $l \subset \mathbb{P}^r$ such that the open subset $T_l = \{L \in T : L \cap l = \emptyset\}$ of T is nonempty. Setting $\mathbb{P}^1 = l$, we define

$$\pi : \tilde{C} \times T_l \rightarrow \mathbb{P}^1 \times T_l$$

mapping a pair (P, L) to the pair $(\pi_L(P), L)$ where by abuse of notation $\pi_L(P)$ is the projection of $f(P)$ from L to l .

By Lemma 4.2 we can find an open dense subset $W \subset T_l$, a curve Y and an integer e , $2 \leq e \leq d/2$, such that for every L in W the morphism π_L factors as

$$\tilde{C} \xrightarrow{\alpha} Y \rightarrow \mathbb{P}^1$$

where α belongs to some irreducible component \mathcal{M} of $\text{Mor}_e(\tilde{C}, Y)$.

Now we claim: if H is a general hyperplane, every degree e effective divisors $E \leq D_H$ is a fibre of some morphism $\alpha \in \mathcal{M}$.

To prove the claim, we use the remark above that there is an open dense subset $A \subset \mathbb{P}^{r*}$ such that for every hyperplane $H \in A$ we can find L in W with $L \subset H$. We may also assume A is contained in the open set $U \subset \mathbb{P}^{r*}$ of hyperplanes transverse to C . By Corollary 4.4 the set

$$I_e(A) = \{(E, H) \in \tilde{C}_e \times U : E \leq D_H, H \in A\}.$$

is irreducible. Let $Z \subset I_e(A)$ be the subset consisting of those (E, H) for which the divisor E is a fibre of some morphism $\alpha \in \mathcal{M}$. The projection $pr_2 : Z \rightarrow A$ is surjective: indeed, for every $H \in A$ we can pick L in W so that $L \subset H$, and π_L factors through some $\alpha \in \mathcal{M}$, hence $D_H = \pi_L^{-1}(H)$ is union of fibres of α . In particular, Z and $I_e(A)$ both have dimension r , hence Z contains an open dense subset Z_0 of $I_e(A)$. Since the projection $pr_2 : I_e(A) \rightarrow A$ is surjective and generically finite, we can find an open dense subset $A_0 \subset A$

such that $pr_2^{-1}(A_0) \subset Z_0 \subset Z$, that is, for $H \in A_0$, every degree e effective divisors $E \leq D_H$ is a fibre of some morphism $\alpha \in \mathcal{M}$, and the claim is proven.

Next we show Y is rational. By way of contradiction, suppose $g(Y) \geq 2$. Then $\text{Mor}_e(\tilde{C}, Y)$ is zero dimensional, so \mathcal{M} consists of a unique morphism α . But then not every degree e subdivisor of D_H can be a fibre of α , hence $g(Y) \leq 1$. On the other hand, if $g(Y) = 1$, then two morphisms in \mathcal{M} differ only by a translation of the elliptic curve Y . In particular, the morphisms in \mathcal{M} have the same fibres, and as above this is impossible. Thus Y must be rational.

Finally we show C is rational. We repeat the monodromy argument above, replacing $I_e(A)$ with the set

$$I_{e,e}(A) = \{(E_1, E_2, H) \in \tilde{C}_e \times \tilde{C}_e \times U : E_1 + E_2 \leq D_H, H \in A\}.$$

The conclusion is that for a general hyperplane H the following holds: for every $(E_1, E_2) \in \tilde{C}_e \times \tilde{C}_e$ such that $E_1 + E_2 \leq D_H$ there is a subspace L in Σ_l such that $L \subset H$, the morphism π_L factors through a morphism $\alpha : \tilde{C} \rightarrow Y \cong \mathbb{P}^1$, and E_1 and E_2 are fibres of α . In particular, any two such divisors E_1 and E_2 are linearly equivalent. Now, if P and Q are distinct points of $C \cap H$, we can find two divisors F_1 and F_2 of degree $e - 1$ such that $P + Q + F_1 + F_2 \leq D_H$. Taking first $(E_1, E_2) = (P + F_1, Q + F_2)$ and then $(E_1, E_2) = (P + F_2, Q + F_1)$ we see that $2P$ and $2Q$ are linearly equivalent on \tilde{C} . Now this holds for a general hyperplane H and every pair of distinct points P and Q in $C \cap H$, so we can find a point $P_0 \in \tilde{C}$ such that $Q - P_0$ is a 2-torsion point in the Jacobian for infinitely many points Q . Therefore \tilde{C} must be rational. \square

Remark 4.5. In the case of \mathbb{P}^3 , the second hypothesis of Theorem 4.1 means Σ is not a Schubert cycle $\sigma(x)$ consisting of the lines through a point $x \in \mathbb{P}^3$. Indeed, given a plane $H \subset \mathbb{P}^3$, if $\sigma(H)$ denote the locus of lines contained in H , it is well known that every irreducible surface $\Sigma \subset \mathbb{G}(1, \mathbb{P}^3)$ intersects $\sigma(H)$ unless $\Sigma = \sigma(x)$ for some $x \in \mathbb{P}^3$.

Corollary 4.6. *Let $C \subset \mathbb{P}^3$ be a reduced irreducible nondegenerate curve. Suppose C is not rational, and $\Sigma \subset \mathbb{G}(1, \mathbb{P}^3)$ is an irreducible closed surface whose general line does not meet C and is decomposable. Then*

$$\Sigma = \sigma(x) = \{L \in \mathbb{G}(1, \mathbb{P}^3) : x \in L\},$$

where $x \in \mathbb{P}^3 \setminus C$ is one of the finitely many non-birational points of C .

Proof. By Theorem 4.1 and the previous remark, we know any such surface is a Schubert cycle $\sigma(x)$ where x is a point of $\mathbb{P}^3 \setminus C$. If projection from x were birational onto its image, then as in the proof of Theorem 3.6 all but finitely many lines in $\sigma(x)$ would be uniform, hence indecomposable. \square

5. SURFACES OF NON-UNIFORM LINES FOR CURVES IN \mathbb{P}^3

In this paragraph we consider the case of *smooth* irreducible curves $C \subset \mathbb{P}^3$. As in Section 3 we let $\mathcal{N}(C)$ denote the locus of lines that do not meet C and are not uniform. and we begin the classification of pairs (C, Σ) where Σ is an irreducible surface in $\mathbb{G}(1, \mathbb{P}^3)$ such that the general line in Σ belongs to $\mathcal{N}(C)$.

By Theorem 3.1 $\mathcal{N}(C)$ has dimension at most two, and we now wish to classify pairs (C, Σ) where Σ is an irreducible surface in $\mathbb{G}(1, \mathbb{P}^3)$ such that the general line in Σ belongs to $\mathcal{N}(C)$.

First of all, as we observed in Remark 3.7, if $x \in \mathbb{P}^3 \setminus C$ is one of the (finitely many) points non birational points for C , then every line through x is decomposable, hence non-uniform. Thus, if $\sigma(x) \cong \mathbb{P}^2$ denotes the Schubert cycle of lines through x , for each such point x the general line in $\sigma(x)$ belongs to $\mathcal{N}(C)$. In fact we will show that, except C is a rational curve of degree three, four or six, these are the only irreducible surfaces in $\mathcal{N}(C)$. To be more precise, we make the following definition.

Definition 5.1. Let C be a nondegenerate irreducible curve in \mathbb{P}^3 . We say C has *special monodromy* if there exists an irreducible Zariski closed surface $\Sigma \subset \mathbb{G}(1, \mathbb{P}^3)$ that is not a Schubert cycle $\sigma(x)$ of lines through a point $x \in \mathbb{P}^3$, with the property: the general line $L \in \Sigma$ does not meet C , and the projection π_L factors as

$$C \xrightarrow{\alpha} \mathbb{P}^1 \xrightarrow{\beta} \mathbb{P}^1$$

where β has degree 2.

We will show in Section 6 that a curve with special monodromy is rational of degree either 4 or 6. In this section we prove:

Theorem 5.2. *Let C be a smooth nondegenerate irreducible curve in \mathbb{P}^3 . Suppose $\Sigma \subset \mathbb{G}(1, \mathbb{P}^3)$ is an irreducible Zariski closed surface such that the general line L in Σ does not meet C and is not uniform. Then one of the following three possibilities holds:*

- (1) *either there exists a non birational point $x \in \mathbb{P}^3$, and $\Sigma = \sigma(x)$ is the cycle of lines through x ; or*
- (2) *C is a twisted cubic curve, and the general line in Σ is the intersection of two osculating planes to C ; or*
- (3) *C is rational and has special monodromy.*

Remark 5.3. The theorem is true more generally for an irreducible nondegenerate curve $C \subset \mathbb{P}^3$ for which the branch divisor B of Proposition 2.7 is reduced. In particular, it holds for curves with only nodes or simple cusps. The proof is the same as below except C^* has to be replaced by B .

In the proof of the Theorem 5.2 we will need the following two lemmas. The first one is elementary and well known:

Lemma 5.4. *Let X be a smooth irreducible projective curve. Suppose $\pi : X \rightarrow \mathbb{P}^1$ is a nonconstant morphism of degree $d \geq 2$. Then π ramifies over at least two distinct points y_1 and y_2 of \mathbb{P}^1 . If there are no other branch points, then X is rational and up to a choice of coordinates π is the map $z \mapsto z^d$.*

The second lemma we need can be proven using a result due to Strano:

Lemma 5.5 (Strano). *Let $X \subset \mathbb{P}^3$ be an arbitrary reduced curve. If $\Sigma \subset \mathbb{G}(1, \mathbb{P}^3)$ is an irreducible Zariski closed surface parametrizing lines that are trisecant to X (meaning $\deg(X.L) \geq 3$), then one of the following holds:*

- (1) either $\Sigma = \sigma(x)$, where x is a point in \mathbb{P}^3 such that $\deg_x(X.L) \geq 3$ for every line L through x ; or
- (2) $\Sigma = \sigma(H)$, where H is a plane in \mathbb{P}^3 containing a subcurve P of X of degree at least 3, and $\sigma(H)$ is the locus of lines contained in H .

Proof. If the projection

$$\rho : J_\Sigma = \{(H, L) \in \mathbb{P}^{3*} \times \Sigma : H \supset L\} \rightarrow \mathbb{P}^{3*}$$

is not dominant, then by Remark 4.5 there is a point $x \in \mathbb{P}^3$ such that $\Sigma = \sigma(x)$, and then every line through x meets X in a scheme of length at least three. This is possible only if $\deg_x(X.L) \geq 3$ for every line $L \in \sigma(x)$.

Suppose now ρ is dominant. This means that the general plane contains a trisecant line $L \in \Sigma$. By [24], Theorem 9, X contains a unique subcurve $P = P(L)$ whose general plane section is $L.X$. Since $\deg(L.X) \geq 3$, P is a plane curve of degree at least 3. Let $H(P)$ be the linear span of P . Then $\sigma(H(P))$ is a surface of trisecant lines for X . Since X contains finitely many subcurves and the general line L of Σ is contained in $\sigma(H(P(L)))$, Σ must be one of the Schubert cycles $\sigma(H(P(L)))$. \square

Proof of Theorem 5.2. Suppose L is a non-uniform line that does not meet C . Then - see Remark 3.1 - the line $\mathbb{P}_L^1 \subseteq \mathbb{P}^{3*}$ dual to L meets the singular locus C_{sing}^* of the dual surface C^* in at least two distinct points.

Suppose first the dual of a general line in Σ meets C_{sing}^* in more than two distinct points. Let X denote the curve C_{sing}^* taken with its reduced scheme structure, and apply Lemma 5.5 to X and to the surface $\Sigma' = \{\mathbb{P}_L^1 : L \in \Sigma\}$. The conclusion is that either there is a point H^* such that $\Sigma' = \sigma(H^*)$, or there is a point $x \in \mathbb{P}^3$ such that $\Sigma = \sigma(x)$. We cannot have $\Sigma' = \sigma(H^*)$ because the general line through H^* in \mathbb{P}^{3*} meets the curve X only at H^* , while we are assuming that a general line in Σ' meets X in more than two points. Thus $\Sigma = \sigma(x)$. Furthermore, as in the proof of 4.6, the point x is not birational.

Suppose now a general line \mathbb{P}_L^1 in Σ' meets X - the singular locus of C^* - in exactly two points H_1^* and H_2^* . If \mathbb{P}_L^1 meets the dual surface C^* only at H_1^* and H_2^* , the branch divisor of π_L is supported at H_1^* and H_2^* . Then, by Lemma 5.4, the curve C is rational and $\pi_L : C \rightarrow \mathbb{P}^1$ is given in some coordinates by $z \mapsto z^d$. Thus L is the intersection of two planes H_1 and H_2 each of them meeting C at a single point with multiplicity d . As L varies in a surface, there must be infinitely many such osculating planes, and this is possible (since we are in char. zero) only if $d = 3$. Thus C is a twisted cubic curve, and Σ consists of those lines that are intersection of two osculating planes to C .

We are left with the case when, for a general L in Σ , the line \mathbb{P}_L^1 meets C_{sing}^* in exactly two points H_1^* and H_2^* , but it also contains a smooth point $H_3^* \in C^*$. Then, by 2.7 and 2.9, the monodromy group of $\pi_L : \tilde{C} \rightarrow \mathbb{P}_L^1$ contains a transposition. Therefore by 2.2 the map π_L factors nontrivially as $\tilde{C} \xrightarrow{\alpha} Y \xrightarrow{\beta} \mathbb{P}^1$. If the map β ramified over more than two distinct points of \mathbb{P}_L^1 , these points would all be singular points of C^* by 2.7 because $\deg(\alpha) \geq 2$, contradicting the assumption that \mathbb{P}_L^1 meet C_{sing}^* in exactly two distinct points. Now Lemma 5.4 implies $Y \cong \mathbb{P}^1$ is rational, and up to a choice of coordinates $\beta : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the map $z \mapsto z^e$ where $e = \deg(\beta) \geq 2$. We cannot have $e \geq 4$ because as above this would imply the existence of infinitely many hyperosculating planes to C . We claim the case $e = 3$

is also impossible. Since $\deg(\alpha) \geq 2$, if we had $e = 3$, we would have either infinitely many planes that osculate C at more than one point, or infinitely many hyperosculating planes. But C has only finitely many hyperosculating planes, while the first alternative contradicts the fact that the map sending a point $P \in \tilde{C}$ to the osculating plane at P is birational. Thus $e = \deg(\beta) = 2$. Finally, if Σ is not a cycle $\sigma(x)$, the curve C must be rational by Corollary 4.6. \square

6. RATIONAL SPACE CURVES ADMITTING SURFACES OF DECOMPOSABLE LINES

In this section we classify smooth curves in \mathbb{P}^3 having special monodromy. The main result is that such a curve is either a rational quartic or a rational sextic.

We begin with some preliminary remarks. Suppose $C \subset \mathbb{P}^3$ is a degree $2d \geq 4$ smooth rational curve with special monodromy. Recall this means there exists an irreducible two dimensional family of lines Σ , not contained in a Schubert cycle $\sigma(x)$, with the property: every line $L \in \Sigma$ in the family does not meet C , and the projection π_L factors as

$$C \xrightarrow{\alpha_L} \mathbb{P}_L^1 \xrightarrow{\beta_L} \mathbb{P}_L^1$$

where β_L has degree 2.

As above, we denote by $V \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d))$ the 4-dimensional subspace defining the embedding $C \subset \mathbb{P}^3 = \mathbb{P}(V^*)$. Since the morphisms β_L have degree two, the vector space V contains a lot of forms g^2 with $g \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. To make this precise, from now on we let $\mathbb{P}_m = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)))$ and look at the embedding $q : \mathbb{P}_d \rightarrow \mathbb{P}_{2d}$ that sends (the class of) a degree d form to (the class of) its square. We will refer to q or to its image $X \subset \mathbb{P}_{2d}$ as the *quadratic Veronese*.

Proposition 6.1. *Let $C \subset \mathbb{P}(V^*)$ be a smooth rational curve of degree $2d \geq 4$ with special monodromy. Then the intersection $X \cap \mathbb{P}(V) \subset \mathbb{P}_{2d}$ has dimension at least one.*

Proof. By definition of special monodromy, the curve C admits a two dimensional family $\Sigma \subset \mathbb{G}(1, \mathbb{P}(V^*))$ of decomposable lines such that, for every $L \in \Sigma$, the projection π_L factors through a degree two morphism $\beta_L : \mathbb{P}^1 \rightarrow \mathbb{P}_L^1$. Up to a choice of coordinates, β_L is the map $z \mapsto z^2$, hence the line $\mathbb{P}_L^1 \subset \mathbb{P}(V) \subset \mathbb{P}_{2d}$ intersects X in at least two points. Thus the surface

$$\Sigma' = \{\mathbb{P}_L^1 \in \mathbb{G}(1, \mathbb{P}(V)) : L \in \Sigma\}$$

is contained in the set of lines secant to $X \cap \mathbb{P}(V)$. Hence $X \cap \mathbb{P}(V)$ is at least one dimensional. \square

Thus, if C has special monodromy, we can find a curve $Y \subset \mathbb{P}_d$ such that $q(Y)$ lies in the three dimensional subspace $\mathbb{P}(V)$ of \mathbb{P}_{2d} , and one expects there are very few such curves. In fact, we show below that Y must either be a conic in \mathbb{P}_2 or a twisted cubic in \mathbb{P}_3 .

Our first remark is that, since C is smooth, we can exclude the case Y is a line:

Lemma 6.2. *Let $C \subset \mathbb{P}(V^*)$ be a smooth rational curve of degree $2d \geq 4$. For any line $Y \subset \mathbb{P}_d$ the conic $q(Y)$ is not contained in $\mathbb{P}(V)$.*

Proof. The linear span of the conic $q(Y)$ is a plane $\mathbb{P}(W) \subset \mathbb{P}(V)$, and the linear series $\mathbb{P}(W)$ defines a projection $C \rightarrow \mathbb{P}^2 = \mathbb{P}(W^*)$ whose image is a conic. Thus C is contained in a quadric cone. But on the quadric cone there are no smooth rational curves of degree greater than 3, for example by [12, Exercise 2.9 p. 384]. \square

We can also say that $q(Y)$ spans $\mathbb{P}(V)$. One way to prove this is to observe:

Lemma 6.3. *The quadratic Veronese $X = q(\mathbb{P}_d) \subset \mathbb{P}_{2d}$ contains no four collinear points.*

Proof. Fix a line $M \subset \mathbb{P}_{2d}$, and think of it as a linear series g_{2d}^1 on \mathbb{P}^1 . Suppose M contains two distinct points of X . Then M corresponds to a subspace $\langle g^2, h^2 \rangle$ of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. It follows that, if M has a base point $p \in \mathbb{P}^1$, then in fact $2p$ is in the base locus of M . We can then remove $2p$ from the base locus and deduce the statement from the case $d-1$ by induction. Therefore we can assume M is base point free and defines a morphism $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $2d$. By Riemann-Hurwitz the ramification divisor has degree

$$\deg(R_\pi) = 4d - 2.$$

Now, if the linear series M contains a divisor $2D$, then $D \subset R_\pi$. Since D has degree d , there are at most 3 such divisors in M , that is, the support of $X \cap M$ consists of at most 3 points (by contrast, M can be tangent to X at one or more points of intersection, and the scheme theoretic intersection $X \cap M$ may well have length 4). \square

Corollary 6.4. *Let $C \subset \mathbb{P}(V^*)$ be a smooth rational curve of degree $2d \geq 4$. Suppose $Y \subset \mathbb{P}_d$ is a curve such that $q(Y) \subset \mathbb{P}(V)$. Then $\mathbb{P}(V)$ is the linear span of $q(Y)$.*

Proof. Let $T = q(Y)$, and note T is a curve in $X \cap \mathbb{P}(V)$. Since $X = q(\mathbb{P}_d)$ is an embedding of \mathbb{P}_d defined by a linear system of quadrics, every curve in X - and in particular T - has even degree. Now T is not a conic by Lemma 6.2, and it cannot be a plane curve of degree ≥ 4 because X contains no 4 collinear points. Thus T spans $\mathbb{P}(V)$. \square

Before proving Y must be a conic or a twisted cubic, we recall results due as far as we know to Hopf and Eisenbud [7, 5.2]. We denote the linear span of a subset $S \subset \mathbb{P}^N$ by the symbol $\langle S \rangle$.

Proposition 6.5. *Let $q : \mathbb{P}_d \rightarrow \mathbb{P}_{2d}$ be the quadratic Veronese.*

(1) (Hopf) *If $H \subset \mathbb{P}_d$ is a linear subspace of dimension r , then*

$$\dim \langle q(H) \rangle \geq 2r.$$

(2) (Eisenbud) *Equality holds if and only if there exist a two dimensional subspace $W \subset \mathbb{P}_a$ and a form $h \in \mathbb{P}_{d-ra}$ such that $H = \mathbb{P}(hS^r(W))$ where $S^r(W)$ is the image of $\text{Sym}^r(W)$ in the space of forms of degree ra - that is, there exist two linearly independent forms F and G of degree a such that $H = \mathbb{P}(\langle hF^r, hF^{r-1}G, \dots, hG^r \rangle)$.*

Proposition 6.6. *Suppose $Y \subset \mathbb{P}_d$ is a reduced irreducible curve of degree e such that $q(Y)$ spans a three dimensional linear subspace $\mathbb{P}(V)$ of \mathbb{P}_{2d} . Suppose further the linear series $\mathbb{P}(V)$ is base point free and defines an embedding $f : \mathbb{P}^1 \rightarrow \mathbb{P}(V^*)$ with image C . Then either $d = e = 2$, or $d = e = 3$ and Y is a twisted cubic. In particular, C is either a rational quartic or a rational sextic.*

Proof. Given a point $s \in \mathbb{P}^1$, we denote by $H(s) \subset \mathbb{P}_d$ the hyperplane of forms vanishing on s . Note that by assumption $\mathbb{P}(V)$ is base point free, hence Y is not contained in a hyperplane $H(s)$. We first prove the following claim:

For general $s \in \mathbb{P}^1$, the intersection of $H(s)$ and Y is transverse, i.e., consists of e distinct points.

Since Y has only finitely many singular points and each of them is contained in only finitely many hyperplanes of the form $H(s)$, the general hyperplane $H(s)$ does not meet the singular locus of Y . Thus if the claim were false, for every s in \mathbb{P}^1 the hyperplane $H(s)$ would be tangent to Y . We contend this implies $Y \subset \Delta$ where Δ is the hypersurface of polynomials having a double root.

Analytic proof: consider the normalization morphism $g : \tilde{Y} \rightarrow Y$ and the correspondence

$$J = \{(x, s) \in \tilde{Y} \times \mathbb{P}^1 : T_x Y \subset H(s)\}.$$

We are assuming the second projection $J \rightarrow \mathbb{P}^1$ is surjective. Since J is one dimensional, we can find near a general point $s \in \mathbb{P}^1$ a local analytic section $s \mapsto (x(s), s)$ of $p_2 : J \rightarrow \mathbb{P}^1$. Choose local coordinates x on \tilde{Y} and s on \mathbb{P}^1 and write $g(x, s) \in \mathbb{C}$ for the value of the polynomial $g(x)$ at s . The condition $T_{x(s)} Y \subset H(s)$ means $g(x(s), s) = \partial_x g(x(s), s) = 0$. Therefore

$$0 = \frac{d}{ds} g(x(s), s) = \partial_x g(x(s), s) x'(s) + \partial_s g(x(s), s) = \partial_s g(x(s), s)$$

so that s is a double root of the polynomial $g(x(s))$. It follows that $Y \subset \Delta$.

Geometric proof: to say for general s the hyperplane $H(s)$ is tangent to Y is to say the hypersurface $Y^* \subset \mathbb{P}^{d*}$ dual to Y contains the rational normal curve

$$\Gamma = \{H(s) : s \in \mathbb{P}^1\}.$$

Since $\Gamma \subset Y^*$, a tangent line L to Γ is contained in some hyperplane Π_L tangent to Y^* , i.e., belonging to Y^{**} . Then $\Pi_L \in Y^{**} \cap \Gamma^* = Y \cap \Delta$, where the last equality follows from the biduality theorem and the fact that Γ is the dual of Δ . Now Π_L has to vary in a one dimensional family because Γ is nondegenerate. Hence $Y \cap \Delta$ is one dimensional, and therefore must coincide with Y .

We have shown $Y \subset \Delta$, that is, every form in Y has at least one root of multiplicity at least 2. It follows the linear series $\mathbb{P}(V)$ spanned by $q(Y)$ contains infinitely many divisors with multiplicity at least 4 at one point. But, since $\mathbb{P}(V)$ is three dimensional, this implies the linear series has infinitely many inflectionary points, a contradiction. Therefore the claim holds: for s general in \mathbb{P}^1 the hyperplane $H(s)$ meets Y in $e = \deg(Y)$ distinct points.

We now want to apply Riemann-Hurwitz to a general tangential projection of C and use the claim to prove that, if e is not small, then the ramification divisor is too large. By a theorem of Kaji [13], for s general in \mathbb{P}^1 , the tangent line $T_{f(s)} C$ meets C only at $f(s)$ and with multiplicity two. Hence projection from $T_{f(s)} C$ is a degree $2d - 2$ morphism $\pi : C = \mathbb{P}^1 \rightarrow \mathbb{P}^1$. In other words, π is the morphism defined by the base point free g_{2d-2}^1 :

$$\{D - 2s : D \in |V|, D \geq 2s\}.$$

Now let g be a form in $Y \cap H(s)$: then $2s$ is a double root of $g^2 \in \mathbb{P}(V)$, hence our g^2_{2d-2} contains the divisor

$$(g^2)_0 - 2s = 2s_1 + \cdots + 2s_{d-1}$$

and we see $s_1 + \cdots + s_{d-1} \leq R_\pi$ where R_π is the ramification divisor of π . Since $Y \cap H(s)$ consists of e distinct points, we conclude $\deg R_\pi \geq e(d-1)$. By Riemann-Hurwitz then

$$-2 = -2(2d-2) + \deg(R_\pi) \geq -2(2d-2) + e(d-1)$$

so that

$$e \leq 4 - \frac{2}{d-1}.$$

Since $e > 1$ (otherwise the span of $q(Y)$ would be a \mathbb{P}^2), we must have either $e = 3$ and $d \geq 3$ or $e = 2$ and $d \geq 2$.

Suppose first $e = 2$. Then Y is a smooth conic spanning a 2-dimensional linear subspace $H \subseteq \mathbb{P}_d$. Since $q|_H : H \rightarrow \mathbb{P}_{2d}$ is defined by a linear system of quadrics, the linear space $\langle q(H) \rangle$ has dimension at most 5, and by Hopf at least 4. If it has dimension 5, then $q|_H$ is the Veronese embedding of $H = \mathbb{P}^2$, and then q maps any conic of \mathbb{P}^2 to a hyperplane section of the Veronese surface, and so $q(Y)$ spans a \mathbb{P}^4 , contrary to our assumptions. Hence $\langle q(H) \rangle$ is a \mathbb{P}^4 .

Now by Eisenbud's result 6.5(2) we have

$$\langle q(H) \rangle = \mathbb{P}(\langle h^2 F^4, h^2 F^3 G, h^2 F^2 G^2, h^2 F G^3, h^2 G^4 \rangle).$$

Thus the linear series $\langle q(H) \rangle$ defines a morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^4$ whose image is a normal rational quartic T , and the linear subseries $\mathbb{P}(V)$ maps \mathbb{P}^1 to a projection C of T in $\mathbb{P}(V^*)$. Therefore $\deg(C) \leq 4$, that is, $d = 2$.

Finally suppose $e = 3$ and $d \geq 3$. First observe Y is a twisted cubic curve. Otherwise Y would be a plane cubic and would have infinitely many trisecant lines. This is impossible since $Y \subset q^{-1}(\mathbb{P}(V))$ and $q^{-1}(\mathbb{P}(V))$ is cut out by quadrics. Thus Y is a twisted cubic curve spanning a three dimensional subspace $H \subset \mathbb{P}_d$. Let $\mathbb{P}^s = \langle q(H) \rangle$. By hypothesis $q(Y)$ spans a \mathbb{P}^3 in \mathbb{P}^s , so Y is contained in $s-3$ linearly independent quadrics. But Y is a twisted cubic curve, thus $h^0(H, \mathcal{I}_{Y,H}(2)) = 3$ and we conclude $s \leq 6$. By Hopf, we must have $s = 6$. As in the case $e = 2$, we can now use Eisenbud's result 6.5(2) to conclude $d = 3$ and C is a rational normal sextic curve. \square

Remark 6.7. The hypothesis $f : \mathbb{P}^1 \rightarrow \mathbb{P}(V^*)$ is an embedding - rather than birational onto its image - is used only to apply the theorem of Kaji (a general tangent to C intersects C only at the point of tangency). But one can avoid using this fact analyzing what happens if the projection from a general tangent line has degree less than $2d-2$.

Remark 6.8. In fact, in both the cases $d = 2$ and $d = 3$, one sees $q(Y) = X \cap \mathbb{P}(V)$ for example using the fact $q^{-1}(X \cap \mathbb{P}(V))$ is cut out by the right number of quadrics. Thus Y is unique, and the surface Σ of decomposable lines for C is determined as the dual of the families of lines in $\mathbb{P}(V)$ that are secant to the rational curve $q(Y)$.

Example 6.9. The proposition is false in positive characteristic. An example in char. 7 of a normal rational quartic $Y \subset \mathbb{P}_4$ such that $q(Y)$ spans only a \mathbb{P}^3 in \mathbb{P}_8 is given by the

parametric equation

$$g(t) = 1 - 2tX - 2t^2X^2 - 4t^3X^3 + 4t^4X^4$$

Indeed $q(Y)$ is parametrized by

$$g(t)^2 = 1 - 4tX + 28t^4X^4 - 32t^7X^7 + 16t^8X^8.$$

Theorem 6.10. *Suppose C is a smooth rational curve having special monodromy. Then $\deg C$ is either 4 or 6. Furthermore, every rational quartic has special monodromy; the general rational sextic does not have special monodromy, but there are rational sextics with special monodromy.*

Proof. Since C has special monodromy, its degree is an even number $2d \geq 4$ and by Proposition 6.1 there is an irreducible curve $Y \subset \mathbb{P}_d$ such that $q(Y) \subset \mathbb{P}(V)$. By Corollary 6.4 the curve $q(Y)$ spans $\mathbb{P}(V)$. Now Proposition 6.6 implies $\deg(C)$ is either 4 or 6.

To see that every rational quartic has special monodromy, we observe the converse of Proposition 6.1 holds: if $C \subset \mathbb{P}(V^*)$ is a smooth rational curve of degree $2d \geq 4$ and $\dim X \cap \mathbb{P}(V) > 0$, then C has special monodromy. Indeed, suppose there is an irreducible curve $T \subset X \cap \mathbb{P}(V)$. By Corollary 6.4, the curve T spans $\mathbb{P}(V)$, that is, it is not contained in a plane. Therefore the secant lines to T are not all contained in one plane. Now let $\Sigma' \subset \mathbb{G}(1, \mathbb{P}(V))$ be the surface of secant lines to T , and let Σ be the corresponding subvariety of $\mathbb{G}(1, \mathbb{P}(V^*))$. Since the secant lines to T are not all contained in one plane, Σ is not contained in a Schubert cycle $\sigma(x)$. Since $\mathbb{P}(V)$ is spanned by T , a general secant line \mathbb{P}_L^1 to T is base point free, so that its dual line $L \in \Sigma$ does not meet C . Furthermore, by construction a general line in Σ is decomposable as prescribed by special monodromy. Therefore C has special monodromy.

Now suppose C is a rational quartic, that is, $d = 2$. Then X is a Veronese surface in \mathbb{P}_4 , and every hyperplane $\mathbb{P}(V)$ in \mathbb{P}_4 cuts a curve on X . Thus C has special monodromy.

When $d = 3$, the quadratic Veronese X has codimension 3 in \mathbb{P}_6 , therefore a general \mathbb{P}^3 in \mathbb{P}_6 does not cut a curve on X , and the general rational sextic does not have special monodromy. For an example of a sextic with special monodromy, consider - cf. [3] - the twisted cubic curve $Y \subset \mathbb{P}^3 \cong \mathbb{C}[X]_{\leq 3}$, image of the embedding $g : \mathbb{P}^1 \rightarrow \mathbb{P}^3$ defined by

$$g(t) = X + tX^3 + t^2 - t^3X^2.$$

One immediately checks $q(Y)$ spans a three dimensional subspace $\mathbb{P}(V)$ in \mathbb{P}_6 , and the linear series $|V|$ defines an embedding of \mathbb{P}^1 in $\mathbb{P}(V^*)$: the image C of this embedding is a rational sextic with special monodromy. This example can be constructed noting $q(Y)$, as every sextic in \mathbb{P}^3 , must have quadrisecant lines, and these must be degenerate because of Lemma 6.3. Thus one looks for Y such that the line joining $g(0)^2 = X^2$ and $g(\infty)^2 = X^4$ is tangent to $q(Y)$ both at $g(0)^2$ and at $g(\infty)^2$. \square

7. EXAMPLES OF CURVES ADMITTING A ONE DIMENSIONAL FAMILY OF NON UNIFORM LINES

We give examples of space curves having a one dimensional family of non uniform lines. The monodromy of all these lines is contained in the alternating group.

We recall the construction given in [22]. Let X be any smooth curve of genus g and L be a spin line bundle on X , i.e. $L^2 = \omega_X$. We fix a point P in X - in fact one could take any effective divisor $D > 0$. Let $V_m = H^0(X, L(mP))$ and $W_m = H^0(X, \omega_X(2mP))$. Given a two plane $\Pi \subset V_m$, we let $S^2(\Pi) \subset W_m$ be the 3-dimensional space image of the natural map $\Pi \otimes \Pi \rightarrow W_m$. If (s, t) is a base of Π , we define a basis of $S^2(\Pi)$ setting $\omega_1 = s^2 - t^2$, $\omega_2 = i(s^2 + t^2)$ and $\omega_3 = 2st$. In [22], a plane Π is called *minimal* if all the elements of $S^2(\Pi)$ are exact meromorphic forms. Then there exist $F_i \in H^0(X, \mathcal{O}_X(2m-1)P)$ such that $dF_i = \omega_i$. The name minimal and the above choice of the basis come from the fact the real parts of the F_i define a map $G : X - P \rightarrow \mathbb{R}^3$

$$G(x) = \text{Re}(F_1(x), F_2(x), F_3(x)),$$

which gives a minimal surface in the Euclidean space. Associated to a minimal plane Π we consider the space:

$$W = \{k \in H^0(X, \mathcal{O}_X(2m-1)P) : dk \in S^2(\Pi)\}.$$

Note that W is generated by the constants and the F_i . If moreover the linear system associated to Π is base point free, then W is also base point free and it defines an immersion $f : X \rightarrow \mathbb{P}^3$ setting $f(x) = (1, F_1, F_2, F_3)$. For suitable choices (for instance if $2m-1$ is a prime number) $f(X)$ is a curve of degree $2m-1$ and hence f is birational onto its image.

The existence, for any X and P , of base point free minimal planes $\Pi \subset V_m$ is proven in [22, Proposition 5.8 p. 355] for $m > 31g + 21$. On the other hand, if $\sigma \in \Pi, \sigma \neq 0$, let $L_\sigma \subset W$ be the two dimensional subspace defined by the equation $dF = \lambda \sigma^2$, $\lambda \in \mathbb{C}$. Note $L_\sigma = \langle 1, F_\sigma \rangle$ where $dF_\sigma = \sigma^2$. Now L_σ defines a non uniform line for $f(X)$. In fact the monodromy of F_σ is contained in the alternating group. This follows because all the ramifications F_σ are odd. The map $\sigma \rightarrow L_\sigma$ defines a $\mathbb{P}^1 = \mathbb{P}(\Pi)$ family of non uniform lines for $C = f(X)$. We remark that all the lines belong to a fixed plane.

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